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Supersymmetric quantum mechanics in quaternion space

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Abstract : Basic formulation of supersymmetry has been carried out in quaternionic Hilbert space in terms of quaternionic spinorial charges and the identification of extra imaginary components with the supertime anti-commuting variables and internal degrees of freedom

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1. Introduction

The idea that the standard quantum mechanics studied in complex Hilbert space may only be an asymptotic version of an underlying nonlinear theory, initially floated by Pearle [1] and then extensively studied although through an empirical model, by Weinberg [2], has now gained immense credence with the evolvement of the theory and experimental observation of the phenomenon of chaos in quantum systems. In this context, quaternions constitute an extremely attractive mechanism for introducing nonlinearities in the hitherto, linear quantum theory [3-8].

In this paper, we have studied Supersymmetric Quantum Mechanics in quaternionic H^3 space which is isomorphic to the twelve dimensional real space $R^6 \times R^6$ and the six dimensional complex space $C^3 \times C^3$. Carrying out the various commutation and anti-commutation relations to be satisfied by the basic variables for the further development of quaternionic supersymmetry, it has been shown that the grading of the Poincare group can be carried out through the identification of the additional imaginary components as the anti-commutation variables. Thus, the SL(2,C) group of complex space has been enhanced to SL(2,H) group which is isomorphic to the SL(4,C) group. Further, the evolution of bosonic and fermionic subspaces in the quaternionic space has been established.

2. Poincare algebra in H^3 space

We define an event A in the quaternionic space H^3 by three quaternionic coordinates

$$A \equiv \left(q^1, q^2, q^3\right) \tag{1}$$

with

$$q^l = x^l + e_A t^l + e_B u^l + e_C v^l, \tag{2}$$

where e_A, e_B, e_C are the quaternion units which satisfy the usual quaternion algebra (which is associative but not commutative).

$$e_A \, e_B = -\delta_{AB} + \sum_{C=1}^{3} \varepsilon_{ABC} \, e_C , \tag{3}$$

with A, B = 1, 2, 3 and where ε_{ABC} is the usual totally antisymmetric three index tensor with $\varepsilon_{123} = 1$.

Further, x, t, u, v are all assumed to be real. Also since we will not be working with complexified quaternions, e_A, e_B, e_C can be conveniently abbreviated as

$$e_A = i, \quad e_B = j, \quad e_C = k.$$

In the natural units, $c = h = 1$ and $x_l = -x^l$, $t_l = t^l$, $u_l = u^l$, $v_l = v^l$. The invariant metric which represents the world interval between two events in the H^3 space, may be written as follows:

$$ds^2 = \sum \mathrm{Re} \, q_l \, q^{l*}, \tag{4}$$

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where Re denotes the real part and * denotes quaternionic conjugation.

The anti-Hermitian generators of space translations $\tilde{P}$ in quaternionic H^3 space are defined by their action in the coordinate representation on an arbitrary state $|f\rangle$ by

$$\langle q|\tilde{P}^a|f\rangle = \frac{\partial}{\partial q^a}\langle q|f\rangle, \tag{5}$$

with the explicit representation:

$$\tilde{P}^a = \tilde{p}^a + i\tilde{E}^a + j\tilde{F}^a + k\tilde{G}^a, \tag{6}$$

where $p = \tilde{\nabla}_x$, $E = \tilde{\nabla}_t$, $F = \tilde{\nabla}_u$, $G = \tilde{\nabla}_v$.

Similarly, the generators of rotations in quaternionic H^3 space may be represented in terms of anti Hermitian operators by

$$\tilde{Z}^a = \tilde{J}^a + i\tilde{K}^a + j\tilde{L}^a + k\tilde{M}^a, \tag{7}$$

where $\tilde{J}$ generates pure rotations $\tilde{K}, \tilde{L}$ and $\tilde{M}$ generate boosts (rotations through imaginary angles) along the relevant axes.

Due to the non-commutativity of the quaternion algebra, the commutator $\left[\tilde{P}^a, \tilde{P}^b\right]$ does not vanish trivially. Further, in order that $\tilde{P}, \tilde{Z}$ generate a representation of the Poincare algebra, the following commutators / anti-commutators need to be satisfied by their components:

$$\left[\tilde{p}^a, \tilde{p}^b\right] = \left[\tilde{E}^a, \tilde{E}^b\right] = \left[\tilde{F}^a, \tilde{F}^b\right] = \left[\tilde{G}^a, \tilde{G}^b\right] = 0, \tag{8}$$

$$\tilde{p}^a, \left(\tilde{E}, \tilde{F}, \tilde{G}\right)^b\Big] = 0, \tag{9}$$

$$\tilde{E}^a, \left(\tilde{F}, \tilde{G}\right)^b\Big\} = 0, \text{ and cyclic permutations thereof,} \tag{10}$$

$$\left[\tilde{J}^a, \left(\tilde{E}, \tilde{F}, \tilde{G}\right)^b\right] = 0, \tag{11}$$

$$\left[\tilde{K}^a, \tilde{E}^b\right] = \tilde{p}^a, \tag{12}$$

$$\left[\tilde{L}^a, \tilde{F}^b\right] = \tilde{p}^a, \tag{13}$$

$$\left[\tilde{M}^a, \tilde{G}^b\right] = \tilde{p}^a, \tag{14}$$

$$\left[\tilde{J}^a, \tilde{p}^b\right] = -\sum_c \varepsilon_{abc}\tilde{p}^c, \tag{15}$$

$$\left[(\tilde{K}, \tilde{L}, \tilde{M})^a, \tilde{p}^b\right] = \delta_{ab}\left(\tilde{E}, \tilde{F}, \tilde{G}\right)^c, \tag{16}$$

$$\tilde{K}^a, \left(\tilde{F}, \tilde{G}\right)^b\Big\} = 0 \text{ and cyclic permutations thereof.} \tag{17}$$

The Hamiltonian for a free particle in H^3 space is given by

$$H = \frac{\tilde{P}^2}{2M} = \frac{\left[\tilde{p}^2 - \left(\tilde{E}^2 + \tilde{F}^2 + \tilde{G}^2\right)\right]}{2M}, \tag{18}$$

where we have assumed the orthogonality of the components of $\tilde{p}$ as represented by the following commutators:

$$\left\{\tilde{p}, \left(\tilde{E}, \tilde{F}, \tilde{G}\right)\right\} = 0$$

and

$\left[\tilde{E}, \left(\tilde{F}, \tilde{G}\right)\right] = 0$ with the cyclic permutations thereof.

The relations

$$[H, P] = 0,$$

$$[H, Z^1] = 0$$

and

$$[H, Z^2] = 0$$

trivially follow from the reality of H.

In order that $\tilde{Z}$ satisfies the SU(2) algebra of angular momentum, the components of $\tilde{Z}$ require to satisfy following identities:

$$\left[\tilde{J}^a, \tilde{J}^b\right] = -\frac{1}{4}\sum_c \varepsilon_{abc}\tilde{J}^c,$$

$$\left[\tilde{J}^a, (\tilde{K}, \tilde{L}, \tilde{M})^b\right] = -\frac{1}{4}\sum_c \varepsilon_{abc}(\tilde{K}, \tilde{L}, \tilde{M})^c,$$

$$\left[\tilde{K}^a, \tilde{K}^b\right] = -\frac{1}{4}\sum_c \varepsilon_{abc}\tilde{J}^c,$$

$$\left[\tilde{L}^a, \tilde{L}^b\right] = -\frac{1}{4}\sum_c \varepsilon_{abc}\tilde{J}^c,$$

$$\left[\tilde{M}^a, \tilde{M}^b\right] = -\frac{1}{4}\sum_c \varepsilon_{abc}\tilde{J}^c,$$

$\left[\tilde{K}^a, \tilde{L}^b\right] = \frac{1}{4}\sum \varepsilon_{abc}\tilde{M}^c$, and its cyclic permutations.

Thus, the operators $\tilde{P}, \tilde{Z}$ and H given by eqs. (6), (7) and (18) respectively constitute the generalized Poincare group in H^3 space.

We now introduce a grading of the above Poincare algebra to get an extended algebra corresponding to the supersymmetric algebra in our quaternionic H^3 Hilbert space. For this purpose we define the quaternionic charges by

$$\tilde{Q}^a = \tilde{q}_0^a + i\,\tilde{q}_1^a + j\tilde{q}_2^a + k\tilde{q}_3^a , \tag{30}$$

here the components of $\tilde{Q}^a$ obey the following commutation relations.

$$\{\tilde{q}_0^a, \tilde{q}_{2,3}^b\} = 0, \tag{31}$$

$[\tilde{q}_1^a, \tilde{q}_{2,3}^b] = 0$ and cyclic permutations thereof. (32)

We then get the graded algebra

$$\{\tilde{q}_0^a, \tilde{q}_0^b\} = -2(\gamma^j k^{-1})\tilde{p}^j, \tag{33}$$

$$\{\tilde{q}_1^a, \tilde{q}_1^b\} = 2i(\gamma^j k^{-1})\tilde{E}^j, \tag{34}$$

$$\{\tilde{q}_2^a, \tilde{q}_2^b\} = 2j(\gamma^j k^{-1})\tilde{F}^j \tag{35}$$

nd

$$\{\tilde{q}_3^a, \tilde{q}_3^b\} = 2k(\gamma^j k^{-1})\tilde{G}^j, \tag{36}$$

that

$$\{\tilde{Q}^a, \tilde{Q}^b\} = -2(\gamma^j k^{-1})\tilde{P}^j, \tag{37}$$

here γ 's are the usual Dirac matrices.

The following further commutators / anti-commutators follow from the commutativity of the grading charge $\tilde{Q}^a$ and the components of the linear momentum:

$$[\tilde{p}^a, \tilde{q}_0^b] = [\tilde{p}^a, \tilde{q}_1^b] = [\tilde{p}^a, \tilde{q}_2^b] = [\tilde{p}^a, \tilde{q}_3^b] = 0, \tag{38}$$

$$\left[(\tilde{E}, \tilde{F}, \tilde{G})^a, \tilde{q}_0^b\right] = \left[(\tilde{E}, \tilde{F}, \tilde{G})^a, \tilde{q}_1^b\right] = \left[(\tilde{E}, \tilde{F}, \tilde{G})^a, \tilde{q}_2^b\right.$$

$$= \left[(\tilde{E}, \tilde{F}, \tilde{G})^a, \tilde{q}_3^b\right] = 0 \tag{39}$$

$\{\tilde{q}^a, \tilde{q}_{2,3}^b\} = 0$ and cyclic permutations thereof. (40)

Our final commutator to complete the graded Lie algebra in quaternionic H^3 space is

$$[\tilde{Q}^a, \tilde{Z}^1] = (\sigma^{jk})^{ab}\,\tilde{Q}^b, \tag{41}$$

which give the following commutation relations for the components of $\tilde{Q}$ and $\tilde{Z}$:

$$[\tilde{q}^a_{(0,1,2,3)}, \tilde{J}^1] = (\sigma^{jk})^{ab}\,\tilde{q}^b_{(0,1,2,3)}, \tag{42}$$

$$\left[\tilde{q}_0^a, (\tilde{K}, \tilde{L}, \tilde{M})^1\right] = (\sigma^{01})^{ab}\,\tilde{q}_0^b, \tag{43}$$

$$[\tilde{q}_1^a, \tilde{K}^1] = -(\sigma^{01})^{ab}\,\tilde{q}_0^b, \tag{44}$$

$$[\tilde{q}_2^a, \tilde{L}^1] = -(\sigma^{01})^{ab}\,\tilde{q}_0^b, \tag{45}$$

$$[\tilde{q}_3^a, \tilde{M}^1] = -(\sigma^{01})^{ab}\,\tilde{q}_0^b, \tag{46}$$

$\left[\tilde{q}_1^a, (\tilde{L}, \tilde{M})^1\right] = (\sigma^{01})^{ab}\,\tilde{q}_0^b$ and its cyclic permutations. (47)

3. Realization of supercharge and superfield in terms of differential operators in H^3 space

In order to obtain the supercharge and superfield in the quaternionic H^3 space, we identify the additional imaginary components (t^b, u^b, v^b) and (t^c, u^c, v^c) as the anti-commuting variables by requiring that

$$\{t^b, t^c\} = \{u^b, u^c\} = \{v^b, v^c\} = 0 . \tag{48}$$

We now construct the following non-Hermitian operators in terms of the anti-commuting variables:

$$Q = Q_A + iQ_B + Q_C + Q_D \tag{49}$$

and

$$\tilde{Q}^\dagger = \tilde{Q}_A^\dagger - i\tilde{Q}_B^\dagger - j\,\tilde{Q}_C^\dagger - \tilde{Q}_D^\dagger , \tag{50}$$

with the additional constraint

$$\tilde{Q}_M^\dagger = -\tilde{Q}_M . \tag{51}$$

In terms of the anti-commuting variables, we take the following representation for the supercharges Q:

$$\tilde{Q}_A = \frac{1}{2}\left(\frac{\partial}{\partial t^b} + \frac{\partial}{\partial u^b} + \frac{\partial}{\partial v^b}\right) - i(\gamma^\mu)t^b\partial_\mu$$

$$-j(\gamma^\mu)u^b\partial_\mu - k(\gamma^\mu)v^b\partial_\mu , \tag{52}$$

$$\tilde{Q}_B = -\frac{1}{2}\frac{\partial}{\partial t^c} + i(\gamma^\mu)t^c\partial_\mu , \tag{53}$$

$$\tilde{Q}_C = -\frac{1}{2}\frac{\partial}{\partial u^c} + j(\gamma^\mu)u^c\partial_\mu \tag{54}$$

and

$$\tilde{Q}_D = -\frac{1}{2}\frac{\partial}{\partial v^c} + k(\gamma^\mu)v^c\partial_\mu \tag{55}$$

with the respective adjoints

$$\tilde{Q}_M^\dagger = -\tilde{Q}_M , \tag{56}$$

where $M = A, B, C, D$.

These operators then satisfy the following relations:

$$\left\{\tilde{Q},\tilde{Q}^{\dagger}\right\}=-2\left[\left(\tilde{Q}_A\right)^2+\left(\tilde{Q}_B\right)^2+\left(\tilde{Q}_C\right)^2+\left(\tilde{Q}_D\right)^2\right.$$

$$=2H,$$

$$[H,\tilde{Q}]=0 \tag{57}$$

and

$$\left[H,\tilde{Q}^{\dagger}\right]=0. \tag{58}$$

We shall constitute our superspace in H^3 space with the variables

$$Z(t,u,v)=\left\{\sum_{i=1}^{3}\left[\left(t^i\right)^2+\left(u^i\right)^2+\left(v^i\right)^2\right]\right\}^{\frac{1}{2}}, \tag{59}$$

$$Z^b(t,u,v)=\left[\left(t^b\right)^2+\left(u^b\right)^2+\left(v^b\right)^2\right]^{\frac{1}{2}} \tag{60}$$

and

$$Z^c(t,u,v)=\left[\left(t^c\right)^2+\left(u^c\right)^2+\left(v^c\right)^2\right]^{\frac{1}{2}}. \tag{61}$$

The superfield can then be formulated as an arbitrary function of these variables or their components and it reduces to the following superposition in the case of $N=1$ quaternion supersymmetry:

$$\begin{aligned}\Im\left(Z,Z^b,Z^c\right)&=q(Z)+it^b\left[\psi(Z)+\overline{\psi(Z)}\right]+ju^b\left[\varphi(Z)+\overline{\varphi(Z)}\right]\\&+kv^b\left[\xi(Z)+\overline{\xi(Z)}\right]+t^c\left[\psi(Z)-\overline{\psi(Z)}\right]+u^c\left[\varphi(Z)-\overline{\phi(Z)}\right]\\&+v^c\left[\xi(Z)-\overline{\xi(Z)}\right]-i\left[t^b,t^c\right]A^t(t)-j\left[u^b,u^c\right]A^u(t)\\&-k\left[v^b,v^c\right]A^v(t),\end{aligned} \tag{62}$$

where the usual position variables $q(t)$ and the function $A^t(t)$ are bosonic and $\psi(Z)$, $\varphi(Z)$ and $\xi(Z)$ are fermionic variables.

4. Supersymmetric transformations in H^3 space

In this space, the supersymmetric transformations are

$$\begin{aligned}Z\to Z'=Z&-i\left\{\left[t^b,\eta^a\right]+\left[t^c,\eta^b\right]\right\}+j\left\{\left[u^b,\upsilon^a\right]+\left[u^c,\upsilon^b\right]\right\}\\&+k\left\{\left[v^b,\tau^a\right]+\left[v^c,\tau^b\right]\right\},\end{aligned} \tag{63}$$

$$t^b\to t'^b=t^b+\eta^a, \tag{64}$$

$$t^c\to t'^c=t^c+\eta^b, \tag{65}$$

$$u^b\to u'^b=u^b+\upsilon^a, \tag{66}$$

$$u^c\to u'^c=u^c+\upsilon^b, \tag{67}$$

$$v^b\to v'^b=v^b+\tau^a \tag{68}$$

and

$$v^c\to v'^c=v^c+\tau^c, \tag{69}$$

where η, υ and τ are the constant anti-commuting variables which constitute the usual anti-commuting parameters as

$$\varepsilon^i=\eta^a+i\eta^b,$$

$$\varepsilon^j=\upsilon^a+j\upsilon^b,$$

$$\varepsilon^k=\tau^a+k\tau^b,$$

$$\bar{\varepsilon}^i=\eta^a-i\eta^b,$$

$$\bar{\varepsilon}^j=\upsilon^a-j\upsilon^b$$

and

$$\bar{\varepsilon}^j=\tau^a-k\tau^b$$

5. Covariant derivatives in H^3 space

The following covariant derivatives may then be constructed in our quaternionic superspace, in conformity with the definition of supercharge:

$$D=\frac{1}{2}\left(\frac{\partial}{\partial t^b}+\frac{\partial}{\partial u^b}+\frac{\partial}{\partial v^b}+i\frac{\partial}{\partial t^c}+j\frac{\partial}{\partial u^c}+k\frac{\partial}{\partial v^c}\right.$$

$$\left.i\left(t^b+it^c\right)\frac{\partial}{\partial t}-j\left(u^b+ju^c\right)\frac{\partial}{\partial u}-k\left(v^b+kv^c\right)\frac{\partial}{\partial v}\right)$$

and

$$\bar{D}=\frac{1}{2}\left(\frac{\partial}{\partial t^b}+\frac{\partial}{\partial u^b}+\frac{\partial}{\partial v^b}-i\frac{\partial}{\partial t^c}-j\frac{\partial}{\partial u^c}-k\frac{\partial}{\partial v^c}\right.$$

$$\left.-i\left(t^b-it^c\right)\frac{\partial}{\partial t}-j\left(u^b-ju^c\right)\frac{\partial}{\partial u}-k\left(v^b-kv^c\right)\frac{\partial}{\partial v}\right)$$

Then we get

$$\{D,\bar{D}\}=-\left\{\tilde{Q},\tilde{Q}^{\dagger}\right\}=-2H,$$

Further,

$$\left[\tilde{Q},\bar{D}\right]=\left[\bar{D},\tilde{Q}^{\dagger}\right]=-2\left(iH_A+jH_B+kH_C\right),$$

where

$$H=H_A+H_B+H_C$$

and

$$H_A = i\frac{\partial}{\partial t},\ H_B = j\frac{\partial}{\partial u},\ H_C = k\frac{\partial}{\partial v}. \tag{81}$$

6. Legrangian and Hamiltonian in H^3 superspace

We can now formulate the Legrangian by using the covariant derivatives D and the superfields φ^i, where i is the intersymmetry index.

For the case $N = 1$, quaternion supersymmetry, we get the following form of the Legrangian density which is invariant in the superspace constituted above under the given supersymmetry transformations (63-74).

$$L = \frac{1}{2}[\bar{D}\varphi][D\varphi] - W(\varphi), \tag{82}$$

where the superpotential $W(\varphi)$ is arbitrary function of the superfield.

The Taylor's expansion of $W(\varphi)$ may be written in the following form in terms of the constituents of our superspace:

$$\begin{aligned} W(\varphi) &= W(q) + \iota^b[W'(q)]_t\left[\psi(Z)+\overline{\psi(Z)}\right] \\ &+ ju^b[W'(q)]_u\left[\varphi(Z)+\overline{\varphi(Z)}\right] + kv^b[W'(q)]_v\left[\xi(Z)+\overline{\xi(Z)}\right] \\ &+ t^c[W'(q)]_t\left[\psi(Z)-\overline{\psi(Z)}\right] + u^c[W'(q)]_u\left[\varphi(Z)-\overline{\varphi(Z)}\right] \\ &+ v^b[W'(q)]_v\left[\xi(Z)-\overline{\xi(Z)}\right] - i\left[t^b, t^c\right][W'(q)]_t A^t(Z) \\ &- j\left[u^b, u^c\right][W'(q)]_u A^u(Z) - k\left[v^b, v^c\right][W'(q)]_v A^v(Z) \\ &+ [W''(q)]_t\left[\overline{\psi(Z)}, \psi(Z)\right] + [W''(q)]_u\left[\overline{\varphi(Z)}, \varphi(Z)\right] \\ &+ [W''(q)]_v\left[\overline{\xi(Z)}, \xi(Z)\right]. \end{aligned} \tag{83}$$

We can then set up the Legrangian in the following form by integration with respect to the anti commuting variables:

$$\begin{aligned} L &= \frac{1}{2}\dot{q}^2 + \frac{1}{2}\left[\left(A^t\right)^2 + \left(A^u\right)^2 + \left(A^v\right)^2\right] + \frac{i}{2}\left(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi}\right) \\ &+ \frac{j}{2}\left(\varphi\dot{\bar{\varphi}} - \dot{\varphi}\bar{\varphi}\right) + \frac{k}{2}\left(\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi}\right) - A^t[W'(q)]_t - A^u[W'(q)]_u \\ &- A^v[W'(q)]_v + \frac{1}{2}\left\{\begin{matrix}[W''(q)]_t[\bar{\psi}, \psi] + [W''(q)]_u[\bar{\varphi}, \varphi] + \\ [W''(q)]_v[\bar{\xi}, \xi]\end{matrix}\right\}. \end{aligned} \tag{84}$$

On using the equations of motion of the Legrangian density

$$\frac{\partial L}{\partial A^i} = A^t - [W'(q)]_t = 0, \tag{85}$$

we get

$$A^t = [W'(q)]_t, \tag{86}$$

so that we can rewrite the Legrangian density in the form:

$$\begin{aligned} L &= \frac{1}{2}\dot{q}^2 + \frac{i}{2}\left(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi}\right) + \frac{j}{2}\left(\varphi\dot{\bar{\varphi}} - \dot{\varphi}\bar{\varphi}\right) + \frac{k}{2}\left(\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi}\right) \\ &- \frac{1}{2}\left\{[W'(q)]_t^2 + [W'(q)]_u^2 + [W'(q)]_v^2\right\} - \frac{1}{2}\left\{[W''(q)]_t[\bar{\psi}, \psi]\right. \\ &\left.+ [W''(q)]_u[\bar{\varphi}, \varphi] + [W''(q)]_v\left[\bar{\xi}, \xi\right]\right\} \end{aligned} \tag{87}$$

The corresponding expression for the total Hamiltonian is then given by

$$\begin{aligned} H &= \frac{1}{2}P^2 + \frac{1}{2}\left\{[W'(q)]_t^2 + [W'(q)]_u^2 + [W'(q)]_v^2\right\} \\ &+ \frac{1}{2}\left\{[W''(q)]_t[\bar{\psi}, \psi] + [W''(q)]_u[\bar{\varphi}, \varphi] + [W''(q)]_v\left[\bar{\xi}, \xi\right]\right\}, \end{aligned} \tag{88}$$

which is easily decomposed into a bosonic part H_B (which does not contain any fermionic degree of freedom) and a fermionic part (which is independent of any bosonic degree of freedom), where

$$H_B = \frac{1}{2}P^2 + \frac{1}{2}\left\{[W'(q)]_t + [W'(q)]_u + [W'(q)]_v\right\} \tag{89}$$

and

$$H_F = \frac{1}{2}\left\{\begin{matrix}[W''(q)]_t[\bar{\psi}, \psi] + [W''(q)]_u[\bar{\varphi}, \varphi] - \\ [W''(q)]_v\left[\bar{\xi}, \xi\right]\end{matrix}\right. \tag{90}$$

Thus, the above formulation is capable of describing systems which classically have both commuting as well as anti-commuting coordinates.

7. Conclusion

The dimensions of H^3 supersymmetry are explained in terms of three real and nine imaginary components. The nine components associated with the quaternion units may be identified in terms of internal degrees of freedom out of which three may be assumed as time coordinates of T^3 space while the other six may be related to purely imaginary degrees of freedom. Thus, the gauge hierarchies or other hypothetical particle structures can be explained better in this quaternionic formulation.

As the quaternionic units describe the curvature in space time, the supersymmetric theory in this space can be integrated with the supergravity theory and its quantization may play an important role towards the understanding of quantum gravity. Besides, through a symplectic decomposition of the quaternionic space into two complex subspaces, the possibility of a unified description of bradyonic and tachyonic bosons and fermions could be explored.

In the context of the nonlinear Schrödinger equation, Weinberg, [2], has made the interesting observation that in general, they correspond to a chaotic dynamics, whereas solutions of the linear time-dependent Schrödinger equation are quasiperiodic. The same observation should apply to the quaternionic case of generalized quantum mechanics which if not unitary, is likely to be chaotic.

Chaotic behaviour in the presence of many degrees of freedom, could provide a mechanism for the emergence of an ensemble exhibiting the probabilistic element associated with the state vector reduction operator R, since in chaotic systems, there is high sensitivity to initial conditions, so neighbouring trajectories could evolve to very different final states.

The requirement for getting standard quantum mechanics from a chaotic deterministic system of equations, has been clearly stated by Pearle [1]. One needs a system in which the possible final states for trajectories are those given by standard complex quantum mechanics, with the probability measure for attaining these states given by the squared modulus of the complex wave function. A possible source of the information distinguishing between the neighbouring trajectories could be the quaternionic cross couplings between the subsystem in question and the rest of the universe which although very small in the asymptotic regime (where they would be the remnants of a hidden very high energy layer of physics), are never precisely zero.

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